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European Journal of Combinatorics 26 (2005) 1033–1052

European Journal of Combinatorics

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Classifying cubic symmetric graphs of order $8p$ or $8p^2$ ☆

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Received 28 January 2004; received in revised form 1 June 2004; accepted 15 June 2004
Available online 3 August 2004

Abstract

A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, we classify the s-regular elementary Abelian coverings of the three-dimensional hypercube for each $s \geq 1$ whose fibre-preserving automorphism subgroups act arc-transitively. This gives a new infinite family of cubic 1-regular graphs, in which the smallest one has order 19 208. As an application of the classification, all cubic symmetric graphs of order $8p$ or $8p^2$ are classified for each prime p, as a continuation of the first two authors' work, in Y.-Q. Feng, J.H. Kwak [Cubic symmetric graphs of order a small number times a prime or a prime square (submitted for publication)] in which all cubic symmetric graphs of order $4p$, $4p^2$, $6p$ or $6p^2$ are classified and of Cheng and Oxley's classification of symmetric graphs of order $2p$, in Y. Cheng, J. Oxley [On weakly symmetric graphs of order twice a prime, J. Combin. Theory B 42 (1987) 196–211].

Keywords: s-regular graphs; s-arc-transitive graphs; Regular coverings

MSC: 05C25; 20B25

☆ Supported by EYTP, NSFC and 985-PBNU in China and Com^{2}MaC-KOSEF in Korea.
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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2004.06.015

1. Introduction

Throughout this paper, we consider a finite connected graph without loops or multiple edges. For a graph X, every edge of X gives rise to a pair of opposite arcs. By $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$, we denote the vertex set, the edge set, the arc set and the automorphism group of the graph X, respectively. The *neighborhood* of a vertex $v \in V(X)$, denoted by $N(v)$, is the set of vertices adjacent to v in X. Let a group G act on a set Ω, and let $\alpha \in \Omega$. We denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing α. The group G is said to be *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$, and *regular* if G is semiregular and transitive on Ω. A graph $\widetilde{X}$ is called a *covering* of X with a projection $p : \widetilde{X} \to X$ if p is a surjection from $V(\widetilde{X})$ to $V(X)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{X}$ is called the *covering graph* and X is the *base graph*. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or K-covering) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purposes of this paper, all functions are composed from left to right). If K is cyclic or elementary Abelian then $\widetilde{X}$ is called a *cyclic* or an *elementary Abelian covering* of X. If $\widetilde{X}$ is connected, K is the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while an element of the covering transformation group fixes each fibre setwise.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A symmetric graph X is said to be *s-regular* if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other. In other words, the automorphism group $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X. Tutte [40,41] showed that every finite cubic symmetric graph is s-regular for some s, and this s is at most five. A subgroup of the automorphism group of a graph is said to be *s-regular* if it acts regularly on the set of s-arcs in the graph.

Djoković and Miller [7] constructed an infinite family of cubic 2-regular graphs, and Conder and Praeger [5] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. Several different kinds of infinite families of tetravalent 1-regular graphs have been constructed in [29,32,39]. The first cubic 1-regular graph was constructed by Frucht [15] and later Miller [36] constructed infinitely many cubic 1-regular graphs of girth 6. From Cheng and Oxley's classification of symmetric graphs of order $2p$ [3], it can be shown that Miller's construction contains the all cubic 1-regular graphs of order $2p$, where $p \ge 13$ is a prime congruent to 1 modulo 3. Marušič and Xu [35] showed a way to construct a cubic 1-regular graph Y from a tetravalent half-transitive graph X with girth 3 by letting the triangles of X be the vertices in Y with two triangles being adjacent when they share a common vertex in X. Using the Marušič and Xu result, Miller's construction can be generalized to graphs of order $2n$, where $n \ge 13$ is odd such that 3 divides $\varphi(n)$, the Euler function (see [1,34]). It may be shown that all cubic 1-regular Cayley graphs on the dihedral groups (see [34]) are exactly those graphs generalized by Miller's construction.

Also, as shown in [33] or [34], one can see the importance of a study of cubic 1-regular graphs in connection with chiral (that is regular and irreflexible) maps on a surface by means of tetravalent half-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus.

Regular coverings of a graph have received considerable attention (see [9–11,13,14,21, 26–28,30,31]). The first two authors classified the s-regular cyclic or elementary Abelian coverings of the complete graph K_4 and the complete bipartite graph $K_{3,3}$ for each $s \geq 1$ in [11,13], whose fibre-preserving subgroups are arc-transitive. As an application of these classifications, all cubic s-regular graphs of order $4p$, $4p^2$, $6p$ or $6p^2$ were constructed for each $s \geq 1$ and each prime p in [13]. Recently, the s-regular cyclic coverings of the three-dimensional hypercube Q_3 were classified by the first and the last authors in [14]. In this paper, we classify the s-regular elementary Abelian coverings of the same base graph Q_3. In particular, we find a new infinite family of cubic 1-regular graphs, in which the smallest one has order 19 208. Each graph in this new family has girth more than 14 and a solvable automorphism group, so this family cannot contain any of the cubic 1-regular graphs discussed in the previous paragraph. It also represents an interesting family of chiral regular maps which cover the Euclidean cube. For more related results, we refer the reader to [20,37,38], where a homological point of view applied to cyclic coverings over Platonic solids is discussed. With the classification of s-regular elementary Abelian coverings of Q_3, we classify the cubic s-regular graphs of order $8p$ or $8p^2$ for each $s \geq 1$ and each prime p. The classification method in this paper cannot be applied to classify the cubic symmetric graphs of order $2p$ or $2p^2$ because some of them cannot be expressed as regular coverings of smaller cubic graphs. In [12], the first two authors used another method to investigate the cubic 1-regular graphs of order twice an odd integer, and classified the cubic 1-regular graphs of order $2p$ and $2p^2$. Note that one can easily classify the cubic s-regular graphs of order $2p$ for each $s \geq 1$ by using Cheng and Oxley's classification of symmetric graphs of order $2p$ [3].

It is well known that a solvable group contains a normal elementary Abelian subgroup. Thus a cubic arc-transitive graph with solvable automorphism group can be reduced by a series of elementary Abelian factorizations to a few small graphs (not necessarily simple). In fact, it was proved in [26] that a cubic graph admitting a solvable edge-transitive group of automorphisms is a regular covering of either the complete graph K_4 of order 4 or a 3-dipole. Note that the common coverings of K_4 and the 3-dipole are the coverings of Q_3 because Q_3 is a common covering of K_4 and the 3-dipole. Hence the classification result in this paper can be used to derive some structural properties of bipartite arc-transitive coverings of K_4 with solvable groups of automorphisms. It will be shown in this paper that a cyclic or an elementary Abelian connected covering of Q_3 with an arc-transitive fibre-preserving subgroup is 1- or 2-regular. A more general statement in [26] implies that regular arc-transitive coverings of Q_3 admitting a solvable arc-transitive group of automorphisms cannot be s-arc-transitive for $s \geq 3$. Recently, Malnič et al. [27] investigated the elementary Abelian coverings of a small cubic graph and obtained a fruitful result explaining what is happening in particular cases including coverings of Q_3. Furthermore, an infinite family of cubic edge- but not vertex-transitive graphs (such graphs are called semisymmetric graphs) was constructed in [28] as cyclic coverings of the bipartite graph $K_{3,3}$.

2. Voltage graphs and lifting problems

Let X be a graph and K a finite group. By a^{-1}, we mean the reverse arc to an arc a. A *voltage assignment* (or K-*voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p : X \times_\phi K \to X$, which is called the *natural projection*. By defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K can be identified with a subgroup of $\mathrm{Aut}(X \times_\phi K)$ acting semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a K-*covering*. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of X with the covering transformation group K can be described as a derived graph $X \times_\phi K$. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be T-*reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [17] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha} p = p\alpha$, we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is, $K = \{\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X}) : p = \widetilde{\alpha} p\}$. Clearly, if $\widetilde{\alpha}$ is a lift of α, then $K\widetilde{\alpha}$ are all the lifts of α.

The problem of whether an automorphism α of X lifts or not can be grasped in terms of voltage as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\overline{\alpha}$ from the set of voltages of fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages of C and C^α, respectively. Note that if K is Abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be replaced by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of [24, Theorem 4.2].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering where ϕ is T-reduced. Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

We have from [24, Corollary 4.3] the following result.

Proposition 2.2. *Let $X \times_\phi K \to X$ be a connected K-covering. Then, an automorphism α of X lifts if and only if, for each closed walk W in X, we have $\phi(W^\alpha) = 1$ if and only if $\phi(W) = 1$.*

For more results on the lifts of automorphisms of X, we refer the reader to [6,25,31].

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2, respectively, are said to be *isomorphic* if there exists a graph isomorphism $\widetilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\widetilde{\alpha} p_2 = p_1$. We quote the following proposition.

Proposition 2.3 (*[18]*). *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where ϕ and ψ are T-reduced, are isomorphic if and only if there exists an automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\phi(u, v)^\sigma = \psi(u, v)$ for any cotree arc (u, v) of X.*

3. Examples and preliminaries

Let m, n be positive integers and let p be a prime. Throughout this paper, we denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and by $\mathbb{Z}_p^m$ the elementary Abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (m times). By Q_3, we denote the three-dimensional hypercube which is bipartite with partite sets $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$. Let K be an Abelian group and let T be a spanning tree of Q_3, as shown by dart lines in Fig. 1. Let ϕ be such a voltage assignment defined by $\phi = 0$ on T and $\phi = z_1, z_2, z_3, z_4$ and z_5 on the cotree arcs $(\mathbf{b}, \mathbf{y})$, $(\mathbf{c}, \mathbf{w})$, $(\mathbf{c}, \mathbf{x})$, $(\mathbf{d}, \mathbf{w})$ and $(\mathbf{d}, \mathbf{x})$ respectively, where 0 is the identity element of K and $z_i \in K$ $(1 \le i \le 5)$. By $X(K; z_1, z_2, z_3, z_4, z_5)$, we denote the derived voltage graph $Q_3 \times_\phi K$, that is the graph with vertex set $V(X(K; z_1, z_2, z_3, z_4, z_5)) = V(Q3) \times K$ and edge set

$$\begin{aligned} E(X(K; z_1, z_2, z_3, z_4, z_5)) = \{&(\mathbf{a}, k)(\mathbf{x}, k), (\mathbf{a}, k)(\mathbf{y}, k), (\mathbf{a}, k)(\mathbf{z}, k), (\mathbf{b}, k)(\mathbf{w}, k),\\ &(\mathbf{b}, k)(\mathbf{z}, k), (\mathbf{c}, k)(\mathbf{z}, k), (\mathbf{d}, k)(\mathbf{y}, k),\\ &(\mathbf{b}, k)(\mathbf{y}, z_1 + k), (\mathbf{c}, k)(\mathbf{w}, z_2 + k), (\mathbf{c}, k)(\mathbf{x}, z_3 + k),\\ &(\mathbf{d}, k)(\mathbf{w}, z_4 + k), (\mathbf{d}, k)(\mathbf{x}, z_5 + k) \mid k \in K\}. \end{aligned}$$

Identity $\mathbb{Z}_p^m$ as the set of row vectors with entries in the field $\mathbb{Z}_p$. Then $\mathbb{Z}_p^m$ is an m-dimensional vector space and z_1, z_2, z_3, z_4, z_5 are row vectors in $\mathbb{Z}_p^m$. Define by M the $5 \times m$ matrix with vector z_i as its i-th row $(1 \le i \le 5)$. Clearly, the graph $X(K; z_1, z_2, z_3, z_4, z_5)$ is uniquely determined by M and we also denote by $X(M)$ the graph $X(K; z_1, z_2, z_3, z_4, z_5)$. Now, we introduce some cubic symmetric graphs that will be used later.

Example 3.1. Let n and k be non-negative integers such that $1 \le k \le n-1$ and $(k, n) = 1$. Let k^{-1} be the inverse of k in $\mathbb{Z}_n^*$. Define

$$CQ_n(k) = X(\mathbb{Z}_n; 1, k, -k^{-1}, -k^{-1} - 1, k)$$

(CQ means the cyclic covering of Q_3). Let p be a prime such that $p - 1$ is divisible by 3. Let $m = p$ or p^2 and let k be an element of order 3 in $\mathbb{Z}_m^*$. Since $\mathbb{Z}_m^*$ is cyclic, there are only two elements of order 3 in $\mathbb{Z}_m^*$, that is, k and k^2. It will be shown in Lemma 3.6 that $CQ_m(k)$ is independent of the choice of the 3-element k in $\mathbb{Z}_m^*$. Thus, we shall denote

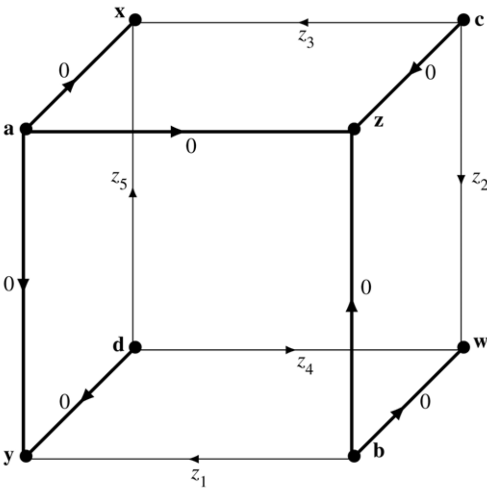

Fig. 1. A spanning tree and a voltage assignment on Q_3.

by CQ_m the graph $CQ_m(k)$. For convenience, we also denote by CQ_2, CQ_3 and CQ_6 the graphs $CQ_2(1)$, $CQ_3(1)$ and $CQ_6(1)$, respectively.

Example 3.2. Let p be a prime and let E be the 5×5 identity matrix with row vectors in $\mathbb{Z}_p^5$. Define

$$EQ_{p^5} = X(E)$$

(EQ means the elementary Abelian covering of Q_3). The graph EQ_{p^5} is well defined because $X(E)$ is only dependent on the dimension 5 of $\mathbb{Z}_p^5$ by Proposition 2.3. Later, in Theorem 4.1, it will be shown that the graph EQ_{p^5} is a cubic 2-regular graph for any prime p.

Example 3.3. Let $p = 3$ or a prime such that $p - 1$ is divisible by 3. Let $\lambda = 1$ when $p = 3$ and let λ be an element of order 3 in $\mathbb{Z}_p^*$ when $p > 3$. Define

$$M(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & -1 & -\lambda & -\lambda \end{bmatrix} \qquad \text{and} \qquad EQ_{p^4}(\lambda) = X(M(\lambda)).$$

Here the row vectors of the matrix $M(\lambda)$ are taken from $\mathbb{Z}_p^4$. It will be shown in Lemma 3.6 that $EQ_{p^4}(\lambda) \cong EQ_{p^4}(\lambda^2)$. Since there are only two elements of order 3 in $\mathbb{Z}_p^*$, the graph $EQ_{p^4}(\lambda)$ is independent of the choice of λ and we denote by EQ_{p^4} the graph $EQ_{p^4}(\lambda)$. By Theorem 4.1, the graph EQ_{3^4} is 2-regular and EQ_{p^4} is 1-regular when $p > 3$ is a prime such that $p - 1$ is divisible by 3.

Example 3.4. Let p be a prime and let

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The row vectors of M_1 or M_2 are taken from $\mathbb{Z}_p^3$ or $\mathbb{Z}_p^2$ respectively. Define

$$EQ_{p^3} = X(M_1) \qquad \text{and} \qquad EQ_{p^2} = X(M_2).$$

It will be shown in Theorem 4.1 that both EQ_{p^3} and EQ_{p^2} are 2-regular for each prime p.

Lemma 3.5. *Let p be a prime such that $p-1$ is divisible by 3, and let $n = p$ or p^2. Then k is an element of order 3 in $\mathbb{Z}_n^*$ if and only if $k^2+k+1 = 0(\bmod n)$.*

Proof. Let $k^2+k+1 = 0(\bmod n)$. Since $p-1$ is assumed to be divisible by 3, $p \geq 7$. If $k = 1(\bmod n)$ then $3 = 0(\bmod n)$, that is, $n = 1$ or 3, which is impossible. Thus, $k \neq 1(\bmod n)$ and since $k^3-1 = (k-1)(k^2+k+1) = 0(\bmod n)$, k is an element of order 3 in $\mathbb{Z}_n^*$.

Conversely, if k is an element of order 3 in $\mathbb{Z}_n^*$, then $k \neq 1(\bmod n)$ and $k^3 = 1(\bmod n)$. It follows that $(k-1)(k^2+k+1) = 0(\bmod n)$. Clearly, if $n = p$ then $k^2+k+1 = 0(\bmod n)$. If $n = p^2$, it suffices to show $(k-1, p^2) = 1$ in order to prove $k^2+k+1 = 0(\bmod p^2)$. Suppose to the contrary that $(k-1, p^2) \neq 1$. Then, $k-1$ is a multiple of p. Let $k = \ell p + 1$. By $k^3 = 1(\bmod p^2)$, we have $(\ell p+1)^3 = \ell^3 p^3 + 3\ell^2 p^2 + 3\ell p + 1 = 1(\bmod p^2)$, implying $3\ell p = 0(\bmod p^2)$. Since $p > 3$, ℓ should be a multiple of p and hence $k = \ell p + 1 = 1(\bmod p^2)$, a contradiction. $\square$

Lemma 3.6. *Let p be a prime such that $p-1$ is divisible by 3. If λ is an element of order 3 in $\mathbb{Z}_p^*$, then $EQ_{p^4}(\lambda) \cong EQ_{p^4}(\lambda^2)$. Let $n = p$ or p^2. If k is an element of order 3 in $\mathbb{Z}_n^*$, then $CQ_n(k) \cong CQ_n(k^2)$.*

Proof. Recall that $V(EQ_{p^4}(\lambda)) = V(EQ_{p^4}(\lambda^2)) = V(Q_3) \times \mathbb{Z}_p^4$ where $\mathbb{Z}_p^4$ is the four-dimensional row vector space over $\mathbb{Z}_p$. To show that $EQ_{p^4}(\lambda) \cong EQ_{p^4}(\lambda^2)$, we define a map α from $V(EQ_{p^4}(\lambda))$ to $V(EQ_{p^4}(\lambda^2))$ by

$$\begin{aligned}
(\mathbf{a}, (i, j, k, \ell)) &\mapsto (\mathbf{a}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{b}, (i, j, k, \ell)) &\mapsto (\mathbf{c}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{c}, (i, j, k, \ell)) &\mapsto (\mathbf{b}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{z}, (i, j, k, \ell)) &\mapsto (\mathbf{z}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{x}, (i, j, k, \ell)) &\mapsto (\mathbf{y}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{y}, (i, j, k, \ell)) &\mapsto (\mathbf{x}, (k+\lambda^2\ell, -j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{w}, (i, j, k, \ell)) &\mapsto (\mathbf{w}, (k+\lambda^2\ell, 1-j, i+\lambda^2\ell, -\lambda\ell)),\\
(\mathbf{d}, (i, j, k, \ell)) &\mapsto (\mathbf{d}, (k+\lambda^2\ell+\lambda^2, 1-j, i+\lambda^2\ell+\lambda^2, \lambda^2-\lambda\ell)).
\end{aligned}$$

For a graph X and $v \in V(X)$, we denote by $N_X(v)$ the neighborhood of v in X in order to emphasize the graph X. Then,

$$N_{EQ_{p^4}(\lambda)}((\mathbf{d}, (i, j, k, \ell))) = \{(\mathbf{w}, (i, j, k, \ell + 1)), (\mathbf{y}, (i, j, k, \ell)), \\ (\mathbf{x}, (i - \lambda, j - 1, k - \lambda, \ell - \lambda))\}.$$

Noting that $\lambda^2 + \lambda + 1 = 0(\bmod\ p)$, we have

$$\begin{aligned} & N_{EQ_{p^4}(\lambda^2)}((\mathbf{d}, (i, j, k, \ell))^\alpha) \\ & = N_{EQ_{p^4}(\lambda^2)}((\mathbf{d}, (k + \lambda^2\ell + \lambda^2, 1 - j, i + \lambda^2\ell + \lambda^2, \lambda^2 - \lambda\ell))) \\ & = \{(\mathbf{w}, (k + \lambda^2\ell + \lambda^2, 1 - j, i + \lambda^2\ell + \lambda^2, -\lambda - \lambda\ell)), \\ & \quad (\mathbf{y}, (k + \lambda^2\ell + \lambda^2, 1 - j, i + \lambda^2\ell + \lambda^2, \lambda^2 - \lambda\ell)), \\ & \quad (\mathbf{x}, (k + \lambda^2\ell, -j, i + \lambda^2\ell, -\lambda\ell))\}. \end{aligned}$$

By the definition of α, we have

$$\{N_{EQ_{p^4}(\lambda)}((\mathbf{d}, (i, j, k, \ell)))\}^\alpha = N_{EQ_{p^4}(\lambda^2)}((\mathbf{d}, (i, j, k, \ell))^\alpha).$$

Similarly, one can show that

$$\begin{aligned} \{N_{EQ_{p^4}(\lambda)}((\mathbf{a}, (i, j, k, \ell)))\}^\alpha &= N_{EQ_{p^4}(\lambda^2)}((\mathbf{a}, (i, j, k, \ell))^\alpha), \\ \{N_{EQ_{p^4}(\lambda)}((\mathbf{b}, (i, j, k, \ell)))\}^\alpha &= N_{EQ_{p^4}(\lambda^2)}((\mathbf{b}, (i, j, k, \ell))^\alpha), \\ \{N_{EQ_{p^4}(\lambda)}((\mathbf{c}, (i, j, k, \ell)))\}^\alpha &= N_{EQ_{p^4}(\lambda^2)}((\mathbf{c}, (i, j, k, \ell))^\alpha). \end{aligned}$$

This implies that α is an isomorphism from $EQ_{p^4}(\lambda)$ to $EQ_{p^4}(\lambda^2)$ because the graphs are bipartite, so $EQ_{p^4}(\lambda) \cong EQ_{p^4}(\lambda^2)$.

Next, we assume that $n = p$ or p^2. Let $i \in \mathbb{Z}_n$ and let β be a map from $CQ_n(k)$ to $CQ_n(k^2)$ defined by

$$\begin{array}{lllllll} (\mathbf{a}, i) & \mapsto & (\mathbf{a}, -i), & \qquad & (\mathbf{b}, i) & \mapsto & (\mathbf{b}, -i - 1), \\ (\mathbf{c}, i) & \mapsto & (\mathbf{d}, -i), & & (\mathbf{d}, i) & \mapsto & (\mathbf{c}, -i), \\ (\mathbf{w}, i) & \mapsto & (\mathbf{w}, -i - 1), & & (\mathbf{x}, i) & \mapsto & (\mathbf{x}, -i), \\ (\mathbf{y}, i) & \mapsto & (\mathbf{z}, -i), & & (\mathbf{z}, i) & \mapsto & (\mathbf{y}, -i). \end{array}$$

Note that $k^2 = k^{-1}(\bmod\ n)$ and by Lemma 3.5, $k^2 + k + 1 = 0(\bmod\ n)$. Then, a similar argument to $EQ_{p^4}(\lambda) \cong EQ_{p^4}(\lambda^2)$ gives rise to $CQ_n(k) \cong CQ_n(k^2)$. $\square$

By Proposition 2.3, one can show that neither $EQ_{p^4}(\lambda)$ and $EQ_{p^4}(\lambda^2)$ nor $CQ_n(k)$ and $CQ_n(k^2)$ are isomorphic as regular coverings of Q_3. By Lemma 3.6, the graphs CQ_p, CQ_{p^2} in Example 3.1 and EQ_{p^4} in Example 3.3 are well defined.

The next proposition shows the s-regularity of cyclic coverings of the hypercube Q_3.

Proposition 3.7 ([14, Theorem 1.1]). *Let $\widetilde{X}$ be a connected cyclic covering of the three-dimensional hypercube Q_3 and let the fibre-preserving subgroup act arc-transitively on $V(\widetilde{X})$. Then, $\widetilde{X}$ is either 1-regular or 2-regular. Furthermore, we have the following:*

(1) $\widetilde{X}$ *is* 1*-regular if and only if* $\widetilde{X}$ *is isomorphic to one of* $CQ_n(k)$ *for* $2 \le k \le n-3$ *satisfying* $n \mid (k^2+k+1)$ *or one of* $CQ_{2n}(2k-1)$ *for* $2 \le k \le n-1$ *satisfying* $n \mid (4k^2-2k+1)$.
(2) $\widetilde{X}$ *is* 2*-regular if and only if* $\widetilde{X}$ *is isomorphic to* Q_3, CQ_2, CQ_3 *or* CQ_6.

By Lemmas 3.5 and 3.6 and Proposition 3.7, we have the following corollary.

Corollary 3.8. *Let* $\widetilde{X}$ *be a connected regular covering of the three-dimensional hypercube* Q_3 *whose covering transformation group is* $\mathbb{Z}_p$ *or* $\mathbb{Z}_{p^2}$ *for a prime* p *and whose fibre-preserving subgroup is arc-transitive. Then, the covering graph* $\widetilde{X}$ *is either* 1*-regular or* 2*-regular. Furthermore,*

(1) $\widetilde{X}$ *is* 1*-regular if and only if* $\widetilde{X}$ *is isomorphic to one of* CQ_p *or* CQ_{p^2}*, where* $p-1$ *is divisible by* 3*.*
(2) $\widetilde{X}$ *is* 2*-regular if and only if* $\widetilde{X}$ *is isomorphic to* Q_3, CQ_2 *or* CQ_3.

The following well-known group result is due to Burnside.

Proposition 3.9 (*[19, Theorem 2.6 of Chapter IV]*). *Let* G *be a finite group and* P *a Sylow* p*-subgroup of* G. *Let* $N_G(P)$ *denote the normalizer of* P *in* G *and* $C_G(P)$ *the centralizer of* P *in* G. *If* $N_G(P) = C_G(P)$*, then* G *has a normal subgroup* N *such that* $G/N \cong P$.

Let X be a cubic connected symmetric graph and G an s-regular subgroup of $\mathrm{Aut}(X)$ for some $s \ge 1$. Let N be a normal subgroup of G and let $\underline{X}$ denote the quotient graph corresponding to the orbits of N. In view of [22, Theorem 9], we have the following proposition.

Proposition 3.10. *If* N *has more than two orbits, then* N *is the kernel of* G *acting on the set of all orbits of* N *and* G/N *is an* s*-regular subgroup of* $\mathrm{Aut}(\underline{X})$. *Furthermore,* X *is a regular covering of* $\underline{X}$ *with the covering transformation group* N.

It is well known that the three-dimensional hypercube Q_3 is 2-regular and $\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$. Let $\alpha = (\mathbf{bcd})(\mathbf{xyz})$, $\beta = (\mathbf{ab})(\mathbf{cd})(\mathbf{wx})(\mathbf{yz})$, $\gamma = (\mathbf{aw})(\mathbf{bx})(\mathbf{cy})(\mathbf{dz})$ and $\delta = (\mathbf{aw})(\mathbf{bx})(\mathbf{cz})(\mathbf{dy})$. From Fig. 1, α is a rotation of the cube around the vertex **a**, β is a rotation around the axis passing through the centers of the face with vertices **a**, **y**, **b** and **z** and its opposite by 180 degrees, and δ is a rotation around the line passing through the centers of the edges **dy** and **cz**. Furthermore, γ is the unique non-trivial automorphism of Q_3 in the center of $\mathrm{Aut}(Q_3)$, and the quotient graph of Q_3 according to the orbits of the normal subgroup generated by γ is the complete graph K_4 of order 4, so S_4 is the full automorphism group of K_4.

Proposition 3.11 (*[14, Lemma 3.2]*). *The full automorphism group* $\mathrm{Aut}(Q_3)$ *of the hypercube* Q_3 *has two* 1*-regular subgroups, that is,* $\langle \alpha, \beta, \gamma \rangle$ *and* $\langle \alpha, \beta, \delta \rangle$. *Furthermore,* $\langle \alpha, \beta, \gamma \rangle \cong A_4 \times \mathbb{Z}_2$, $\langle \alpha, \beta, \delta \rangle \cong S_4$ *and* $\mathrm{Aut}(Q_3) = \langle \alpha, \beta, \gamma, \delta \rangle$.

Lemma 3.12. *Let* $\widetilde{X}$ *be a connected regular covering of a cubic* 2*-regular graph* X *with a finite transformation group* K. *Suppose that* $\mathrm{Aut}(X)$ *lifts and let* B *be the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *lifted by* $\mathrm{Aut}(X)$. *If* K *is characteristic in* B *or normal in* $\mathrm{Aut}(\widetilde{X})$ *then* $\widetilde{X}$ *is* 2*-regular.*

Proof. Let $A = \mathrm{Aut}(\widetilde{X})$. Since $\mathrm{Aut}(X)$ is 2-regular, B is 2-regular and since $B \leq A$, $\widetilde{X}$ is 2-arc-transitive. It suffices to show that $A = B$. Suppose to the contrary that $A \neq B$. Then, A is at least 3-regular and by Djoković and Miller [7, Theorem 3], A is 3-regular because it contains the 2-regular subgroup B. It follows that $|A : B| = 2$ and so $B \lhd A$. If K is characteristic in B then $K \lhd A$. Thus, we always have $K \lhd A$ by the hypothesis. By Proposition 3.10, A/K is a 3-regular subgroup of $\mathrm{Aut}(X)$. This is impossible because X is 2-regular. □

For a graph Y, we let $Y^{(2)}$ denote the canonical double covering of Y, that is the graph obtained by assigning the voltage $1 \in \mathbb{Z}_2$ to every arc of Y.

Lemma 3.13. *Let Y be a connected vertex-transitive graph. Then, $Y^{(2)}$ is connected if and only if Y is not bipartite. Furthermore, if Y is s-arc-transitive then so is $Y^{(2)}$.*

Proof. If Y is not bipartite, then there is an odd cycle passing through some vertex, say u. Thus, there is a path from $(u, 0)$ to $(u, 1)$. By the connectivity and transitivity of Y, $Y^{(2)}$ is connected. If Y is bipartite we denote by U and V its two partite sets. Let $U_0 = \{(u, 0) \mid u \in U\}$ and $V_1 = \{(v, 1) \mid v \in V\}$. Clearly, the induced subgraph $\langle U_0, V_1\rangle$ of $U_0 \cup V_1$ in $Y^{(2)}$ is a union of some connected components of $Y^{(2)}$. Thus, $Y^{(2)}$ is not connected. Now, assume that Y is s-arc-transitive. By Proposition 2.2, every automorphism of Y lifts, implying that $Y^{(2)}$ is also s-arc-transitive. □

4. Elementary Abelian coverings of the hypercube Q_3

In [14], the first and the last authors classified the cyclic coverings of the three-dimensional hypercube Q_3. In this section, we shall classify the elementary Abelian coverings of the same graph Q_3.

Theorem 4.1. *Let p be a prime and let $\widetilde{X}$ be a connected regular covering of the three-dimensional hypercube Q_3, whose covering transformation group is cyclic or elementary Abelian and whose fibre-preserving subgroup is arc-transitive. Then, $\widetilde{X}$ is either 1-regular or 2-regular. Furthermore, if the transformation group is cyclic then*

(1) *$\widetilde{X}$ is 1-regular if and only if $\widetilde{X}$ is isomorphic to one of $CQ_n(k)$ for $2 \leq k \leq n-3$ satisfying $n \mid (k^2+k+1)$ or one of $CQ_{2n}(2k-1)$ for $2 \leq k \leq n-1$ satisfying $n \mid (4k^2-2k+1)$,*
(2) *$\widetilde{X}$ is 2-regular if and only if $\widetilde{X}$ is isomorphic to Q_3, CQ_2, CQ_3 or CQ_6.*

If the transformation group is an elementary Abelian group $\mathbb{Z}_p^m$ $(m \geq 2)$ then

(3) *$\widetilde{X}$ is 1-regular if and only if $\widetilde{X}$ is isomorphic to one of EQ_{p^4}, where $p-1$ is divisible by 3,*
(4) *$\widetilde{X}$ is 2-regular if and only if $\widetilde{X}$ is isomorphic to either one of EQ_{p^2}, EQ_{p^3} and EQ_{p^5}, or EQ_{3^4}.*

Proof. By Proposition 3.7, we only consider the case when the transformation group is an elementary Abelian group $\mathbb{Z}_p^m$ $(m \geq 2)$. Let $\widetilde{X} = Q_3 \times_\phi \mathbb{Z}_p^m$ $(m \geq 2)$ be a covering graph of Q_3 satisfying the hypotheses in the theorem, where $\phi = 0$ on a spanning tree T

Table 1
Fundamental cycles and their images with corresponding voltages

C	$\phi(C)$	C^{α}	$\phi(C^{\alpha})$	C^{β}	$\phi(C^{\beta})$
azby	z_1	**axcz**	$-z_3$	**byaz**	z_1
bzcw	z_2	**cxdw**	$z_3 + z_4 - z_2 - z_5$	**aydx**	z_5
azcx	z_3	**axdy**	$-z_5$	**bydw**	$z_1 + z_4$
aydwbz	z_4	**azbwcx**	$z_3 - z_2$	**bzcxay**	$z_3 - z_1$
aydx	z_5	**azby**	z_1	**bzcw**	z_2
C	$\phi(C)$	C^{γ}	$\phi(C^{\gamma})$	C^{δ}	$\phi(C^{\delta})$
azby	z_1	**wdxc**	$z_2 + z_5 - z_3 - z_4$	**wcxd**	$z_3 + z_4 - z_2 - z_5$
bzcw	z_2	**xdya**	$-z_5$	**xcza**	$-z_3$
azcx	z_3	**wdyb**	$-z_1 - z_4$	**wczb**	$-z_2$
aydwbz	z_4	**wczaxd**	$z_4 - z_2 - z_5$	**wdyaxc**	$z_2 - z_3 - z_4$
aydx	z_5	**wczb**	$-z_2$	**wdyb**	$-z_1 - z_4$

and we assign voltages z_1, z_2, z_3, z_4 and z_5 to the cotree arcs $(\mathbf{b}, \mathbf{y})$, $(\mathbf{c}, \mathbf{w})$, $(\mathbf{c}, \mathbf{x})$, $(\mathbf{d}, \mathbf{w})$ and $(\mathbf{d}, \mathbf{x})$ respectively, as shown in Fig. 1. Note that $V(Q_3) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and $z_i \in \mathbb{Z}_p^m$. Since $Q_3 \times_\phi \mathbb{Z}_p^m$ is connected, we get $\langle z_1, z_2, z_3, z_4, z_5 \rangle = \mathbb{Z}_p^m$. By Proposition 3.11, the automorphism group $\mathrm{Aut}(Q_3)$ has two 1-regular subgroups $\langle \alpha, \beta, \gamma \rangle$ and $\langle \alpha, \beta, \delta \rangle$, where $\alpha = (\mathbf{bcd})(\mathbf{xyz})$, $\beta = (\mathbf{ab})(\mathbf{cd})(\mathbf{wx})(\mathbf{yz})$, $\gamma = (\mathbf{aw})(\mathbf{bx})(\mathbf{cy})(\mathbf{dz})$, and $\delta = (\mathbf{aw})(\mathbf{bx})(\mathbf{cz})(\mathbf{dy})$. By the hypotheses, the fibre-preserving subgroup, say $\widetilde{L}$, of the covering graph $Q_3 \times_\phi \mathbb{Z}_p^m$ acts arc-transitively on $Q_3 \times_\phi \mathbb{Z}_p^m$. Hence, the projection, say L of $\widetilde{L}$, is arc-transitive on the base graph Q_3. Thus, $L = \langle \alpha, \beta, \gamma \rangle$, $\langle \alpha, \beta, \delta \rangle$ or $\mathrm{Aut}(Q_3)$. It follows that $\alpha, \beta \in L$, and so α and β lift to automorphisms of $Q_3 \times_\phi \mathbb{Z}_p^m$.

Denote by $i_1 i_2 \cdots i_s$ a directed cycle which has vertices $i_1, i_2, \ldots, i_s$ in a consecutive order. There are five fundamental cycles **azby**, **bzcw**, **azcx**, **aydwbz** and **aydx** in Q_3, which are generated by the five cotree arcs $(\mathbf{b}, \mathbf{y})$, $(\mathbf{c}, \mathbf{w})$, $(\mathbf{c}, \mathbf{x})$, $(\mathbf{d}, \mathbf{w})$ and $(\mathbf{d}, \mathbf{x})$, respectively. Each cycle is mapped to a cycle of the same length under the actions of α, β, γ and δ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of Q_3 and $\phi(C)$ denotes the voltage of C.

Note that $\mathbb{Z}_p^m$ is viewed as a vector space over $\mathbb{Z}_p$. Then, an automorphism of the group $\mathbb{Z}_p^m$ can be viewed as a linear transformation of the vector space. Consider the mapping $\overline{\alpha}$ from the set $\{z_1, z_2, z_3, z_4, z_5\}$ of voltages of the five fundamental cycles of Q_3 to the group $\mathbb{Z}_p^m$, which is defined by $\phi(C)^{\overline{\alpha}} = \phi(C^{\alpha})$, where C ranges over the five fundamental cycles. From Table 1, one can see that $z_1^{\overline{\alpha}} = -z_3$, $z_2^{\overline{\alpha}} = z_3 + z_4 - z_2 - z_5$, $z_3^{\overline{\alpha}} = -z_5$, $z_4^{\overline{\alpha}} = z_3 - z_2$ and $z_5^{\overline{\alpha}} = z_1$. In a similar way, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ can be defined and their values can be found easily from Table 1. Since $\alpha, \beta \in L$, Proposition 2.1 implies that $\overline{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of $\mathbb{Z}_p^m$. We denote by α^* and β^* these extended automorphisms, respectively.

If $z_1 = 0$ then $z_1^{\alpha^*} = 0$. By Table 1, $z_3 = -z_1^{\alpha^*} = 0$. Also, $z_5 = 0$ and $z_2 = 0$ because $z_3^{\alpha^*} = -z_5$ and $z_5^{\beta^*} = z_2$. Thus, $\langle z_1, z_2, z_3, z_4, z_5 \rangle = \langle z_4 \rangle = \mathbb{Z}_p^m$, contrary to the fact that $m \geq 2$. Similarly, if $z_2 = 0$ one can get the same contradiction. Consequently, $z_1 \neq 0$ and $z_2 \neq 0$. Now, we claim that z_1 and z_2 are linearly independent. Suppose to the contrary that z_2 is a scalar multiple of z_1, say $z_2 = kz_1$. Then, $k \neq 0$. Since $z_2^{\beta^*} = kz_1^{\beta^*}$, we have $z_5 = kz_1$. As $z_5^{\alpha^*} = kz_1^{\alpha^*}$, z_3 is a scalar multiple of z_1, say $z_3 = \ell z_1$. And, z_4 is also a scalar multiple of z_1 because $z_3^{\beta^*} = \ell z_1^{\beta^*}$. It follows that $\langle z_1, z_2, z_3, z_4, z_5\rangle = \langle z_1\rangle = \mathbb{Z}_p^m$, a contradiction. Thus, z_1 and z_2 are linearly independent. Similarly, z_1 and z_3 are also linearly independent.

Now, we consider four cases: $K = \mathbb{Z}_p^m$ $(m \geq 5)$, $\mathbb{Z}_p^2$, $\mathbb{Z}_p^3$ or $\mathbb{Z}_p^4$, separately.

Case I. $K = \mathbb{Z}_p^m$ $(m \geq 5)$

In this case, $\langle z_1, z_2, z_3, z_4, z_5\rangle = K$ implies that $m = 5$, so z_1, z_2, z_3, z_4, z_5 are linearly independent. From Table 1, one may check that $\phi((\mathbf{azby})^\tau), \phi((\mathbf{bzcw})^\tau), \phi((\mathbf{azcx})^\tau), \phi((\mathbf{aydwbz})^\tau), \phi((\mathbf{aydx})^\tau)$ are linearly independent for $\tau = \alpha, \beta, \gamma$ or δ. Thus, $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^5$ and by Proposition 2.1, α, β, γ and δ lift to automorphisms of $Q_3 \times_\phi \mathbb{Z}_p^5$. Since $\mathrm{Aut}(Q_3)$ is 2-regular and $\mathrm{Aut}(Q_3) = \langle \alpha, \beta, \gamma, \delta\rangle$, $\mathrm{Aut}(Q_3)$ lifts and $\mathrm{Aut}(Q_3 \times_\phi \mathbb{Z}_p^5)$ contains a 2-regular subgroup, say B, lifted by $\mathrm{Aut}(Q_3)$. Clearly, $\mathbb{Z}_p^5 \lhd B$. By Proposition 2.3, we may assume that $z_1 = (1, 0, 0, 0, 0)$, $z_2 = (0, 1, 0, 0, 0)$, $z_3 = (0, 0, 1, 0, 0)$, $z_4 = (0, 0, 0, 1, 0)$ and $z_5 = (0, 0, 0, 0, 1)$. By the definition of the graph EQ_{p^5} in Example 3.2, it is isomorphic to the covering graph $Q_3 \times_\phi \mathbb{Z}_p^5$. Now, we need to show that $Q_3 \times_\phi \mathbb{Z}_p^5$ is 2-regular.

Since B is 2-regular, $|B| = 48p^5$ and $Q_3 \times_\phi \mathbb{Z}_p^5$ is 2-arc-transitive. First, assume $p \geq 5$. Since $p \nmid 48$, $\mathbb{Z}_p^5 \lhd B$ implies $\mathbb{Z}_p^5 \lhd\lhd B$, which means that $\mathbb{Z}_p^5$ is characteristic in B. By Lemma 3.12, $Q_3 \times_\phi \mathbb{Z}_p^5$ is 2-regular. Secondly, assume $p = 3$. Let $O_3(B)$ be the largest normal 3-subgroup of B. Then, $\mathbb{Z}_3^5 \leq O_3(B)$. Suppose $|O_3(B)| = 3^6$. By Proposition 3.10, $B/O_3(B)$ is 2-regular on the quotient graph corresponding to the orbits of $O_3(B)$, which is impossible because $|B/O_3(B)| = 16$ is not divisible by 3. Thus, $|O_3(B)| = 3^5$ and $O_3(B) = \mathbb{Z}_3^5$, that is, K is characteristic in B. By Lemma 3.12, $Q_3 \times_\phi \mathbb{Z}_3^5$ is 2-regular. Lastly, assume $p = 2$. By Conder and Dobcsányi [4], there is no cubic connected 3-arc-transitive graph of order 256, and by the 2-arc-transitivity of $Q_3 \times_\phi \mathbb{Z}_2^5$, it is 2-regular.

Case II. $K = \mathbb{Z}_p^2$

First, let $p \geq 3$. Since z_1 and z_2 are linearly independent and $\langle z_1, z_2, z_3, z_4, z_5\rangle = K$, z_3, z_4 and z_5 can be expressed as a combination of z_1 and z_2. Let $z_5 = kz_1 + \ell z_2$, where the scalars k and ℓ belong to $\mathbb{Z}_p$. Thus, $z_5^{\beta^*} = kz_1^{\beta^*} + \ell z_2^{\beta^*}$. By Table 1, we have $z_2 = (k + k\ell)z_1 + \ell^2 z_2$ which, in view of the linear independence of z_1 and z_2, implies the following equations:

$$k(\ell + 1) = 0, \tag{1}$$

$$\ell^2 = 1. \tag{2}$$

In the rest of this section, as shown in Eqs. (1) and (2), all equations only containing the scalars in $\mathbb{Z}_p$ are to be taken modulo p and the symbol mod p is always omitted. This should cause no confusion. Let $z_3 = mz_1 + nz_2$.

By Eq. (2), we have $\ell = 1$ or -1. Suppose $\ell = -1$. Then, $z_5 = kz_1 - z_2$. Since $z_3^{\beta^*} = mz_1^{\beta^*} + nz_2^{\beta^*}$, we have $z_1 + z_4 = mz_1 + nz_5$ by Table 1. Consequently, $z_4 = (m - 1 + nk)z_1 - nz_2$. As $z_5^{\alpha^*} = kz_1^{\alpha^*} - z_2^{\alpha^*}$, $z_1 = z_2 - (k + 1)z_3 - z_4 + z_5$, and by replacing z_3, z_4 and z_5 with their combinations of z_1 and z_2, the linear independence of z_1 and z_2 implies the following equations:

$$m(k+2)+k(n-1)=0, \tag{3}$$
$$nk=0. \tag{4}$$

Similarly, we have the following equation from $z_3^{\alpha^*}=mz_1^{\alpha^*}+nz_2^{\alpha^*}$ by considering the coefficient of z_2:

$$mn+1=0. \tag{5}$$

Note that $p\neq 2$. If $n\neq 0$ then $k=0$ by Eq. (4) and $m=0$ by Eq. (3). This implies that we always have $mn=0$. By Eq. (5), $1=0$, a contradiction.

Thus, we have $\ell=1$. By Eq. (1), $2k=0$ and so $k=0$. It follows that $z_5=z_2$. As $z_5^{\alpha^*}=z_2^{\alpha^*}$, $z_3+z_4=z_1+2z_2$. Since $z_3^{\beta^*}=mz_1^{\beta^*}+nz_2^{\beta^*}$, we have $z_1+z_4=mz_1+nz_5=mz_1+nz_2=z_3$, that is, $z_3-z_4=z_1$. Noting that $z_3+z_4=z_1+2z_2$, we have $z_3=z_1+z_2$ and $z_4=z_2$. By Proposition 2.3, we may assume that

$$z_1=(1,0),\quad z_2=z_4=z_5=(0,1),\quad z_3=(1,1). \tag{6}$$

From Table 1, it is easy to check that $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^2$. By Proposition 2.1, α, β, γ and δ lift; that is, $\mathrm{Aut}(Q_3)$ lifts. Thus, $\mathrm{Aut}(Q_3\times_\phi\mathbb{Z}_p^2)$ contains a 2-regular subgroup, say B, lifted by $\mathrm{Aut}(Q_3)$. Clearly, $|B|=48p^2$. By the definitions of EQ_{p^2} in Example 3.4 and the equations in (6), $Q_3\times_\phi\mathbb{Z}_p^2$ is isomorphic to EQ_{p^2}. Note that we have assumed that $p\neq 2$. But, for $p=2$ we also have that $Q_3\times_\phi\mathbb{Z}_2^2$ is 2-arc-transitive, where the voltage assignment ϕ corresponds to the equations in (6). By Conder and Dobcsányi [4], there is only one cubic connected symmetric graph of order 32, which is 2-regular. This implies that the $\mathbb{Z}_2^2$-coverings of Q_3 are isomorphic to EQ_{2^2}. Similarly, the $\mathbb{Z}_3^2$-coverings of Q_3 are isomorphic to EQ_{3^2}. Now, we may assume $p\geq 5$. In this case, $\mathbb{Z}_p^2\lhd B$ implies $\mathbb{Z}_p^2\lhd\lhd B$. By Lemma 3.12, EQ_{p^2} is 2-regular.

Case III. $K=\mathbb{Z}_p^3$

Let $p\geq 3$. Note that z_1 and z_3 are linearly independent. Now, we claim that z_1, z_3 and z_4 are linearly dependent.

Suppose to the contrary that z_1, z_3 and z_4 are linearly independent. Since $K=\mathbb{Z}_p^3$, z_5 and z_2 are combinations of z_1, z_3 and z_4. Let $z_5=iz_1+jz_3+kz_4$. Since $z_5^{\beta^*}=iz_1^{\beta^*}+jz_3^{\beta^*}+kz_4^{\beta^*}$, $z_2=(i+j-k)z_1+kz_3+jz_4$ by Table 1. As $z_5^{\alpha^*}=iz_1^{\alpha^*}+jz_3^{\alpha^*}+kz_4^{\alpha^*}$, $z_1=-kz_2+(k-i)z_3-jz_5$ and by replacing z_2 and z_5 with their combinations of z_1, z_3 and z_4, the linear independence of z_1, z_3 and z_4 implies the following equations:

$$k(k-i-j)=1+ij, \tag{7}$$
$$k^2=k-i-j^2, \tag{8}$$
$$2jk=0. \tag{9}$$

Remark that $z_2=(i+j-k)z_1+kz_3+jz_4$. Since $z_2^{\alpha^*}=(i+j-k)z_1^{\alpha^*}+kz_3^{\alpha^*}+jz_4^{\alpha^*}$, $z_3+z_4-z_1-z_2=-(i+j-k)z_3-kz_5+j(z_3-z_2)$ and by replacing z_2 and z_5 with their combinations of z_1, z_3 and z_4 and by considering the coefficient of z_4, we have the following equation:

$$j(j-1)+k(k-1)+1=0. \tag{10}$$

Since $p \neq 2$, $jk=0$ by Eq. (9), that is, $j=0$ or $k=0$. If $j=0$ then $k^2-k+1=0$ by Eq. (10) and $k^2-k+i=0$ by Eq. (8). It follows that $i=1$. By Eq. (7), $k^2-k-1=0$ and so $2=0$ because $k^2-k+1=0$. This is impossible because $p \geq 3$. If $k=0$ then $j^2-j+1=0$ by Eq. (10). By Eq. (8), $i=-j^2$ and by Eq. (7), $j^3=1$. Multiplying j by the equation $j^2-j+1=0$, we have $j^2-j-1=0$. Thus, we get the same contradiction $2=0$.

Now, we have proved that z_1, z_3 and z_4 are linearly dependent. We may assume $z_4=iz_1+jz_3$. Suppose that z_5 is a combination of z_1 and z_3. As $z_4^{\alpha^*}=iz_1^{\alpha^*}+jz_3^{\alpha^*}$, z_2 would be a combination of z_1 and z_3. Consequently, $\mathbb{Z}_p^3=\langle z_1,z_2,z_3,z_4,z_5\rangle=\langle z_1,z_3\rangle$, a contradiction. Thus, z_1, z_3 and z_5 are linearly independent.

As $z_4^{\alpha^*}=iz_1^{\alpha^*}+jz_3^{\alpha^*}$, $z_2=(i+1)z_3+jz_5$. Since $z_2^{\beta^*}=(i+1)z_3^{\beta^*}+jz_5^{\beta^*}$, we have $z_5=(i+1)(z_1+z_4)+jz_2$ and by replacing z_2 and z_4 with their combinations of z_1, z_3 and z_5, the linear independence of z_1, z_3 and z_5 implies that $j^2=1$ and $i=-1$. It follows that $i=-1$ and either $j=1$ or -1.

If $j=1$ then $z_2=z_5$ and $z_4=-z_1+z_3$. As $z_2^{\alpha^*}=z_5^{\alpha^*}$, $z_1=z_3+z_4-z_2-z_5=2z_3-z_1-2z_5$ by Table 1. The linear independence of z_1, z_3 and z_5 implies $2=0$, a contradiction.

Thus, we have $j=-1$. It follows that $z_2=-z_5$ and $z_4=-z_1-z_3$. By Proposition 2.3, we may assume that $z_1=(1,0,0)$, $z_3=(0,1,0)$, $z_5=(0,0,1)$, $z_2=(0,0,-1)$ and $z_4=(-1,-1,0)$. In this case, it is easy to check that $Q_3\times_\phi\mathbb{Z}_p^3$ is isomorphic to EQ_{p^3} in Example 3.4, and that $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, $\overline{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^3$. Thus, $\mathrm{Aut}(Q_3\times_\phi\mathbb{Z}_p^3)$ contains a 2-regular subgroup, say B, lifted by $\mathrm{Aut}(Q_3)$. This is also true for $p=2$. Note that we have assumed that $p\neq 2$. By Conder and Dobcsányi [4], there is only one cubic connected symmetric graph of order 64, which is 2-regular. Consequently, the $\mathbb{Z}_2^3$-coverings of Q_3 are isomorphic to EQ_{2^3}. Also, there is no cubic connected 3-arc-transitive graph of order 216, so the $\mathbb{Z}_3^3$-coverings of Q_3 are isomorphic to EQ_{3^3}. Now, we may assume $p\geq 5$. Then, $\mathbb{Z}_p^3\lhd B$ implies $\mathbb{Z}_p^3 \lhd\lhd B$ and by Lemma 3.12, $Q_3\times_\phi\mathbb{Z}_p^3$ is 2-regular.

Case IV. $K=\mathbb{Z}_p^4$

Recall that z_1 and z_2 are linearly independent. If z_3 is a combination of z_1 and z_2 then, by considering the action of β^* on the combination, z_4 is a combination of z_1 and z_5 by Table 1. It follows that $\mathbb{Z}_p^4=\langle z_1,z_2,z_5\rangle$, a contradiction. Thus, z_1, z_2 and z_3 are linearly independent. Suppose $z_4=iz_1+jz_2+kz_3$. If $j=0$ then z_2 is a combination of z_3 and z_5 because $z_4^{\alpha^*}=iz_1^{\alpha^*}+kz_3^{\alpha^*}$ and so $\mathbb{Z}_p^4$ can be generated by z_1, z_3 and z_5, a contradiction. Therefore, $j\neq 0$ and by $z_4^{\beta^*}=iz_1^{\beta^*}+jz_2^{\beta^*}+kz_3^{\beta^*}$, z_5 is a combination of z_1, z_2 and z_3. Consequently, $\mathbb{Z}_p^4$ can be generated by z_1, z_2 and z_3, a contradiction. Thus, z_1, z_2, z_3 and z_4 are linearly independent. Let $z_5=iz_1+jz_2+kz_3+\ell z_4$.

Since $z_5^{\beta^*}=iz_1^{\beta^*}+jz_2^{\beta^*}+kz_3^{\beta^*}+\ell z_4^{\beta^*}$, $z_2=(i+k-\ell)z_1+\ell z_3+kz_4+jz_5$ and by replacing z_5 with its combination, the linear independence of z_1, z_2, z_3 and z_4 implies the following equations:

$$i + k - \ell + ij = 0, \tag{11}$$
$$j^2 = 1, \tag{12}$$
$$\ell + jk = 0, \tag{13}$$
$$k + j\ell = 0. \tag{14}$$

Similarly, by $z_5^{\alpha^*} = iz_1^{\alpha^*} + jz_2^{\alpha^*} + kz_3^{\alpha^*} + \ell z_4^{\alpha^*}$, we have

$$i(j + k) + 1 = 0, \tag{15}$$
$$j^2 + j + \ell + jk = 0, \tag{16}$$
$$\ell + j - i - k^2 - jk = 0, \tag{17}$$
$$j - j\ell - k\ell = 0. \tag{18}$$

Suppose $p = 2$. By Eqs. (12) and (15), $j = 1$ and $i = 1$, and by Eq. (18), $\ell = 1$ and $k = 0$. Then, Eq. (17) implies $1 = 0$, a contradiction.

Thus, we have $p \geq 3$. By Eq. (12), $j = 1$ or -1. If $j = 1$ then $k+\ell = 0$ by Eq. (13) and $2 = 0$ by Eq. (16), contrary to the fact that $p \geq 3$. Thus, $j = -1$. By Eqs. (13) and (18), $k = \ell$ and $k^2 - k + 1 = 0$, and by Eq. (17), $i = k$. It follows that $i = k = \ell$ and $j = -1$.

If $p = 3$ then $k^2 - k + 1 = 0$ implies $k = -1$ and we have $i = j = \ell = -1$. By Proposition 2.3 we may assume $z_1 = (1, 0, 0, 0)$, $z_2 = (0, 1, 0, 0)$, $z_3 = (0, 0, 1, 0)$, $z_4 = (0, 0, 0, 1)$ and $z_5 = (-1, -1, -1, -1)$. From Table 1, it is easy to see that $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ can be extended to automorphisms of $\mathbb{Z}_3^4$, and by Proposition 2.1, $Q_3 \times_\phi \mathbb{Z}_3^4$ is at least 2-regular. Clearly, $Q_3 \times_\phi \mathbb{Z}_3^4 \cong EQ_{3^4}$ and by Conder and Dobcsányi [4], there is no cubic connected 3-arc-transitive graph of order 648, so EQ_{3^4} is 2-regular.

Now, we consider $p \geq 5$. If $k = -1$, $k^2 - k + 1 = 0$ implies $3 = 0$, a contradiction. Thus, $k \neq -1$. Since $k^2 - k + 1 = 0$, we have $k^3 = -1$ and so $-k$ is an element of order 3 in $\mathbb{Z}_p^*$, say $-k = \lambda$. It follows that $i = k = \ell = -\lambda$ and $j = -1$. Thus, $p - 1$ is divisible by 3 and hence $p \geq 7$. By Proposition 2.3, we may let $z_1 = (1, 0, 0, 0)$, $z_2 = (0, 1, 0, 0)$, $z_3 = (0, 0, 1, 0)$, $z_4 = (0, 0, 0, 1)$ and $z_5 = (-\lambda, -1, -\lambda, -\lambda)$. Clearly, the covering graph $Q_3 \times_\phi \mathbb{Z}_p^4$ is isomorphic to EQ_{p^4} in Example 3.3. From Table 1, it is easy to see that $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ can be extended to automorphisms of $\mathbb{Z}_p^4$, but $\overline{\delta}$ cannot. Thus, EQ_{p^4} is arc-transitive. We claim that EQ_{p^4} is actually 1-regular.

Let $A = \mathrm{Aut}(Q_3 \times_\phi \mathbb{Z}_p^4)$ and suppose to the contrary that $Q_3 \times_\phi \mathbb{Z}_p^4$ is s-regular for some $s \geq 2$. By Tutte [40,41], $s \leq 5$ and so $|A|$ is a divisor of $48 \cdot 8p^4$. Let P be a Sylow p-subgroup of A. Then, $P \cong \mathbb{Z}_p^4$ because $p \geq 7$ and so $P = K$. Let B be the 1-regular subgroup of A lifted by $\langle \alpha, \beta, \gamma \rangle$. Then, $P \lhd B$ and $|B| = 24p^4$. It follows that $B \leq N_A(P)$ and $|A : N_A(P)| \mid 16$. By Sylow's theorem, the number of Sylow p-subgroups of A is $np+1$ and $np+1 = |A : N_A(P)|$. Thus, $(np+1) \mid 16$. Since $p \geq 7$, we have either $np+1 = 1$, or $p = 7$ and $n = 1$. If $np + 1 = 1$ then $P \lhd A$, implying that A/P is an s-regular subgroup of $\mathrm{Aut}(Q_3)$ ($s \geq 2$). Since $\mathrm{Aut}(Q_3)$ is 2-regular, $A/P = \mathrm{Aut}(Q_3)$ and so $\mathrm{Aut}(Q_3)$ lifts, contrary to the fact that $\overline{\delta}$ cannot lift. If $p = 7$ and $n = 1$ then P is not normal in A and $|A| = 8|N_A(P)|$. As $B \leq N_A(P)$, $|A|$ has a divisor $8 \cdot 24p^4$ and so $Q_3 \times \mathbb{Z}_p^4$ is at least 4-regular. Let $H = N_A(P)$. Then, $|A : H| = 8$. By considering the right multiplication action of A on the set of right cosets of H in A, $|A/H_A| \mid 8!$, where H_A is the largest normal subgroup of A in H. Let L be a Sylow 7-subgroup of H_A. As $P \lhd H$, $L \lhd H_A$, implying that L is characteristic in H_A. Thus, $H_A \lhd A$ implies $L \lhd A$. Since P is not normal in A, $L \neq P$ and since $|A/H_A| \mid 8!$, $7^3 \mid |H_A|$. Consequently, $|L| = 7^3$. By Proposition 3.10, the quotient graph corresponding to the orbits of L on $V(Q_3 \times \mathbb{Z}_p^4)$ is a cubic connected s-regular graph of order 56 for some $s \geq 4$. This is impossible by Conder and Dobcsányi [4].

Remark. From the proof of Theorem 4.1, one can see that if $\widetilde{X}$ is 1-regular then $\langle \alpha, \beta, \gamma \rangle \cong A_4 \times \mathbb{Z}_2$ lifts and if $\widetilde{X}$ is 2-regular then $\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$ lifts. In both cases, $\mathrm{Aut}(\widetilde{X})$ is solvable. It follows that $\mathrm{Aut}(EQ_{p^4}) \cong \mathbb{Z}_p^4 \rtimes (A_4 \times \mathbb{Z}_2)$ where $p-1$ is divisible by 3 and that $\mathrm{Aut}(EQ_{p^2}) \cong \mathbb{Z}_p^2 : (S_4 \times \mathbb{Z}_2)$, $\mathrm{Aut}(EQ_{p^3}) \cong \mathbb{Z}_p^3 : (S_4 \times \mathbb{Z}_2)$, $\mathrm{Aut}(EQ_{p^5}) = \mathbb{Z}_p^5 : (S_4 \times \mathbb{Z}_2)$ and $\mathrm{Aut}(EQ_{3^4}) \cong \mathbb{Z}_3^4 : (S_4 \times \mathbb{Z}_2)$, where group extensions are also split if $p \geq 5$. From Theorem 4.1, one can construct an infinite family of cubic 1-regular graphs of type EQ_{p^4}, where $p-1$ is divisible by 3. Since 7 is the smallest prime p such that $p-1$ is divisible by 3, the smallest cubic 1-regular graph in this family has order 19 208. By the 1-regularity of these graphs, one may obtain infinitely many chiral regular maps which cover the Euclidean cube (see [8]).

5. The cubic symmetric graphs of order $8p$ or $8p^2$

In this section, we shall classify the cubic symmetric graphs of order $8p$ or $8p^2$, where p is a prime. If a cubic symmetric graph of order $8p$ is not connected then its components have order 4, 8, $2p$ or $4p$; of these, the first two are the complete graph K_4 and the three-dimensional hypercube Q_3 and the last two have been considered in [3] and [13], respectively. Thus, we only consider the connected ones.

There is a cubic connected symmetric graph of order 28 discovered by Coxeter and investigated by Tutte [42]. For its construction, see Biggs [2, Fig. 2(ii)]; and we denote this graph by C_{28}. By Biggs [2], C_{28} is 3-regular and non-bipartite. Denote by D_{20} the dodecahedron of order 20. The dodecahedron D_{20} is 2-regular and non-bipartite. By Lemma 3.13, the canonical double coverings $C_{28}^{(2)}$ and $D_{20}^{(2)}$ of C_{28} and D_{20} are connected 3-arc-transitive and 2-arc-transitive, respectively. In fact, they are both 3-regular by Conder and Dobcsányi's list of cubic connected s-regular graphs of order up to 768 in [4]. Note that D_{20} is 2-regular and $D_{20}^{(2)}$ is 3-regular, which implies that some automorphisms of $D_{20}^{(2)}$ are not fibre-preserving, being considered as the canonical double-covering graph of D_{20}.

There is a cubic connected 2-regular graph of order 56 discovered by Lorimer. It is a coset graph of the group PSL(2, 7) with respect to a given subgroup of order 3. For the concept of coset graph and the construction of this graph, see Lorimer [22,23]. We denote by L_{56} this cubic 2-regular graph of order 56.

Theorem 5.1. *Let p be a prime and let X be a cubic connected symmetric graph of order $8p$. Then, X is* 1*-regular,* 2*-regular or 3-regular. Furthermore,*

(1) *X is* 1*-regular if and only if X is isomorphic to one of CQ_p, where $p-1$ is divisible by* 3;

(2) *X is* 2*-regular if and only if X is isomorphic to CQ_2, CQ_3 or the Lorimer graph L_{56} of order* 56;

(3) *X is 3-regular if and only if X is isomorphic to the canonical double covering $D_{20}^{(2)}$ or $C_{28}^{(2)}$.*

Proof. Let X be a cubic connected symmetric graph of order $8p$ for a prime p and let $A = \mathrm{Aut}(X)$. Then, by Tutte [41], X is at most 5-regular. Thus, $|A|$ is a divisor of $48 \cdot 8p$. To prove the theorem, we repeatedly use the Conder and Dobcsányi's list of cubic connected s-regular graphs of order up to 768 in [4] and Corollary 3.8.

For example, if $p = 2$, there is one and only one cubic connected symmetric graph of order 16 in the Conder and Dobcsányi's list, and it should be the 2-regular graph CQ_2 by Corollary 3.8. For a similar reason, CQ_3 is the only cubic 2-regular graph of order 24 (for $p = 3$), and $D_{20}^{(2)}$ is the only cubic 3-regular graph of order 40 (for $p = 5$). For $p = 7$, there are three cubic connected symmetric graphs of order 56, which are the 1-regular graph CQ_7, the 2-regular graph L_{56} and the 3-regular graph $C_{28}^{(2)}$. Similarly, for each prime $p = 11, 17, 23, 29, 41, 47, 53, 59, 71$ or 83 (i.e., $8p = 88, 136, 184, 232, 328, 376, 424, 472, 568$ or 664, respectively), there is no connected cubic symmetric graph of order $8p$. For each prime $p = 13, 19, 31, 37, 43, 61, 67, 73$ or 79 (i.e., $8p = 104, 152, 248, 296, 344, 488, 536, 584$ or 632, respectively), there is only one cubic connected symmetric graph of order $8p$ from the list in [4], which is the 1-regular graph CQ_p by Corollary 3.8. Thus, one may assume that $p \geq 89$.

Let P be a Sylow p-subgroup of A with $p \geq 89$. To complete the proof, it suffices to show that $P \lhd A$, because then X is a regular covering of the hypercube Q_3 with the covering transformation group $P \cong Z_p$ by Proposition 3.10 and so $X \cong CQ_p$ with $p-1$ being divisible by 3 by Corollary 3.8. Denote by $N_A(P)$ the normalizer of P in A. By Sylow's theorem, the number of Sylow p-subgroups of A is $np+1$ and $|A : N_A(P)| = np+1$, which divides $48 \cdot 8 = 384$. Suppose $np+1 > 1$. Then, $np+1 = 96, 128, 192$ or 384 because $np+1 \geq 90$. If $np+1 = 96$ then $np = 95 = 5 \times 19$, which is impossible because $p \geq 89$. Thus, $n = 1$ and $p = 127, 191$ or 383. If $p = 383$ then $N_A(P) = P$, implying $C_A(P) = P$, where $C_A(P)$ is the centralizer of P in A. By Proposition 3.9, A has a normal subgroup N such that $A/N \cong P$, and by Proposition 3.10, the quotient graph corresponding to the orbits of N has odd order and valency 3, a contradiction. It follows that $p = 127$ or 191. As $|A| = (np+1)|N_A(P)|$, $|A|$ is divisible by $8 \cdot 2^4 p$ or $8 \cdot 3 \cdot 2^3 p$, implying that X is at least 4-regular. Noting that $|A|$ is a $\{2, 3, p\}$-group, we have that A is solvable because there is no simple $\{2, 3, p\}$-group for $p = 127$ or 191 by Gorenstein [16, pp. 12–14]. Thus, any minimal normal subgroup of A is elementary Abelian. Consider the quotient graph of X corresponding a given minimal normal subgroup of A. Then, the quotient graph is a cubic connected symmetric graph with order $4 \times 127 = 508$, $2 \times 127 = 254$, $4 \times 191 = 764$, $2 \times 191 = 382$, 4 or 8. By Proposition 3.10, the quotient graph is at least 4-regular, which is impossible by Conder and Dobcsányi [4]. Thus, $np+1 = 1$ and so $P \lhd A$, as required. □

Theorem 5.2. *Let p be a prime and let X be a cubic connected symmetric graph of order $8p^2$. Then, X is either 1-regular or 2-regular. Furthermore,*

(1) *X is 1-regular if and only if X is isomorphic to one of CQ_{p^2} where $p-1$ is divisible by 3.*

(2) *X is 2-regular if and only if X is isomorphic to one of EQ_{p^2}.*

Proof. For each prime $p = 2$, 3 or 5, by Conder and Dobcsányi [4], there exists one and only one cubic connected symmetric graph of order $8p^2$, which is actually the cubic 2-regular graph EQ_{p^2} (Theorem 4.1), and for $p = 7$ there are two cubic connected symmetric graphs of order 392, which are the 1-regular graph CQ_{7^2} (Corollary 3.8) and the 2-regular graph EQ_{7^2} (Theorem 4.1). Thus, we may assume $p \geq 11$.

Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. Then, $P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$ and $|A : N_A(P)| = np + 1$, where $N_A(P)$ is the normalizer of P in A. To prove the theorem, it suffices to show that $P \lhd A$ by Corollary 3.8, Proposition 3.10 and Theorem 4.1. Suppose to the contrary that P is not normal in A. Then, $np + 1 \geq 12$ because $p \geq 11$. If $N_A(P) = P$ then $C_A(P) = N_A(P) = P$, where $C_A(P)$ is the centralizer of P in A. By Proposition 3.9, A has a normal subgroup N such that $A/N \cong P$, and by Proposition 3.10, the quotient graph corresponding to the orbits of N has odd order and valency 3, a contradiction. Thus, we assume $N_A(P) \neq P$. Since X is at most 5-regular, $|A| \mid (48 \cdot 8p^2)$ and so $(np + 1) \mid 2^7$ or $3 \cdot 2^6$. It follows that np is one of the following: $15 = 3 \times 5, 31, 63 = 3^2 \times 7, 127, 11, 23, 47, 95 = 5 \times 19, 191$. Since $p \geq 11$, there are two cases: either $n = 1$ and $p = 11, 23, 31, 47, 127, 191$, or $n = 5$ and $p = 19$.

Case I. $n = 1$ and $p = 11, 23, 31, 47, 127$ or 191

Let $H = N_A(P)$. By considering the right multiplication action of A on the set of right cosets of H in A, $|A/H_A| \mid (p + 1)!$, where H_A is the largest normal subgroup of A in H. This forces $p \mid |H_A|$. Let L be a Sylow p-subgroup of H_A. Clearly, L is characteristic in H_A and so $L \lhd A$. Since the Sylow p-subgroups of A are not normal, $p^2 \nmid |H_A|$. Thus, $L \cong \mathbb{Z}_p$. By Proposition 3.10, the quotient graph $\underline{X}$ of X corresponding to the orbits of L is a cubic connected symmetric graph of order $8p$. By Theorem 5.1, $p \neq 11, 23, 47$ or 191. Thus, $p = 31$ or 127 and hence $|A| = 8 \cdot 4 \cdot |N_A(P)|$ or $8 \cdot 2^4 \cdot |N_A(P)|$, implying that X is at least 3-regular. By Proposition 3.10, the quotient graph $\underline{X}$ of X corresponding to the orbits of L is a cubic connected symmetric graph of order 8×31 or 8×127, and it is at least 3-regular. This is impossible by Theorem 4.1.

Case II. $p = 19$ and $n = 5$

In this case, $|A : N_A(P)| = 96$ and $|A| = 8 \cdot 12|N_A(P)|$, implying that X is at least 3-regular. By Gorenstein [16, pp. 12–14], A is solvable. Let N be a minimal normal subgroup of A and let $\underline{X}$ be the quotient graph of X corresponding to the orbits of N. Then, N is elementary Abelian, and the quotient graph has order $8, 2 \times 19 = 38, 4 \times 19 = 76, 8 \times 19 = 152, 2 \times 19^2 = 722$ or 4×19^2. By Proposition 3.10, $\underline{X}$ is at least 3-regular and by Conder and Dobcsányi [4], this is impossible if the quotient graph $\underline{X}$ has order 8, 38, 76, 152 or 722. Now, we may assume that $\underline{X}$ has order 4×19^2. Since A/N is a subgroup of $\mathrm{Aut}(\underline{X})$ that is at least 3-regular, the quotient graph of $\underline{X}$ corresponding to the orbits of a given minimal normal subgroup of A/N is at least 3-regular with order 4, 38, 76 or 722, which is impossible by Conder and Dobcsányi's list in [4].

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